Introduction

The theory of motives is an attempt to formulate a "universal Weil cohomology theory" for projective varieties (recall a Weil cohomology theory is a contravariant functor from SmProj_k to F-Vect, char $F = 0$, satisfying the usual Künneth-formula, cup product, Hard Lefschetz, etc.). This is to some extent successful, but many hard foundational questions remain.

On the other hand a more recent development is the theory of derived categories of sheaves on projective varieties. In many ways derived categories appear motivic in nature: Functors between them are induced by objects on the product (just like correspondences) and they admit decompositions (semiorthogonal decompositions) which in many cases looks like direct sum decompositions of motives. More recent work has indicated the possible existence of a deep relationship between derived categories and birational geometry.

Motivated by this, it is reasonable to think of $D^b(X)$ as a "noncommutative algebraic variety". However there are technical reasons why one should avoid the setting of triangulated categories, so for us we will always work w/ some dg-enhancement or ∞-enhancement (the latter with localizing invariants). So then one should view an arbitrary dg-category as a "noncommutative algebraic variety". The theory of noncommutative motives is then to provide some "universal invariant" of dg-categories.

Let us now sketch the outline of the seminar:

1) dg-categories and derived categories of varieties.
1.5) Review of pure motives.
2) Noncommutative motives
3) Additive invariants
4) Localizing invariants.

Lets try to provide some motivation for (1). To any quasi-compact and separated scheme X, one can give a triangulated category $D^b(\mathrm{coh}\,X)$ $(= D^b_{\mathrm{coh}}(\mathrm{Qcoh}\,X))$ which originally was used to study vector bundles and such and their relation to the geometry of X. This category is typically constructed in three steps. First one passes to the category of chain complexes, $C^b(\mathrm{coh}\,X)$, then to the homotopy category by quotienting out the morphisms homotopic to zero, then finally one applies a localization process on $K^b(\mathrm{coh}\,X)$ to invert quasi-isomorphisms. The resulting category is called the bounded derived category of coherent sheaves $D^b(X)$. As we alluded to earlier, this category appears to exhibit motivic behavior. For example if X and Y are smooth and projective then a fully faithful triangulated functor $\Phi : D^b(X) \to D^b(Y)$ is "represented" by an object on the product, that is,

$$\Phi(-) \cong Rp_*(q^*(-) \overset{L}{\otimes} \mathcal{E}) \quad (\text{"of Fourier-Mukai type"})$$

where $\mathcal{E} \in D^b(X \times Y)$ and $X \xleftarrow{q} X\times Y \xrightarrow{p} Y$. This "pull-multiply-push" bears a striking resemblance to the morphisms (correspondences) in $\mathrm{Mot}_k^{\sim}$. However not all functors are of this type, and to fix this we pass to dg-categories, and for technical reasons look at the category of perfect objects $\mathrm{Perf}\,X \subset D^b(X)$ consisting of bounded complexes of locally free sheaves.

To any scheme as above we regard $\mathrm{Perf}\,X$ as a "noncommutative algebraic variety". Morphisms between noncommutative varieties are exactly the Fourier-Mukai functors. Then $K_0(\mathrm{Perf}\,X)$ plays the role of the Chow ring, and there are so-called additive invariants which are the analog of Weil cohomology theories.

Moreover these additive invariants turn certain decompositions of Perf X into direct sums, and further the similar question of if there is a "universal invariant" has an affirmative answer, which is one of our goals in this seminar.

Finally, there is no apriori reason why we stick to noncommutative varieties of the form Perf X (these come from commutative varieties after all), and so we should actually work with arbitrary dg-categories.

Differential Graded Categories

Let $C(k)$ be the category of chain complexes of k-vector spaces.

Def: A differential graded (dg) category is a category enriched over $C(k)$. A dg-functor is a functor enriched over $C(k)$, i.e. if $\Phi: \mathcal{A} \to \mathcal{B}$ is a dg-functor, then for all objects $c, d \in A$, the morphism

$$\mathrm{Hom}(c,d) \longrightarrow \mathrm{Hom}(\Phi(c), \Phi(d))$$

is a morphism in $C(k)$.

Of course a definition is useless without examples, so let us give a handful of them

Examples:

1) Recall a dg k-algebra is a graded algebra $A = \bigoplus_{i \in \mathbb{Z}} A^i$ together with a differential, i.e. a degree $+1$ morphism $d: A \to A$ whose square is zero and d satisfies the Leibniz rule. Then a simple rephrasing of this is to say that a dg-algebra A is the same as a dg-category $\mathcal{A}$ with one object, and $\mathrm{Hom}_{\mathcal{A}}(*, *) = A$.

2) A left (or right) dg-module over a dg-algebra A is a chain complex M equipped with a chain map $A \otimes M \to M$ which satisfies $d_M(\rho(a,v)) = \rho(d_A a, v) + (-1)^{|a|} \rho(a, d_M v)$ and $\rho(a, \rho(b,v)) = \rho(a \cdot b, v)$. Alternatively given the dg-category $\mathcal{A}$ with one object and endomorphism algebra A, a left (right) dg-module over $\mathcal{A}$ is a dg-functor $\mathcal{A} \to C_{dg}(k)$ ($\mathcal{A}^{op} \to C_{dg}(k)$). Hence more generally for an arbitrary dg-category $\mathcal{A}$, we can define the category of right dg-mods over $\mathcal{A}$ as all dg-functors $\mathcal{A}^{op} \to C_{dg}(k)$.

   Given an object a of $\mathcal{A}$, there is a canonical dg-module given by $\mathrm{Hom}_{\mathcal{A}}(-, a)$. Any dg-module isomorphic to such a functor is called <u>representable</u>.

3) $C(k)$ - the category of complexes of k-modules is a dg-cat, $C_{dg}(k)$, in a natural way. Given two complexes $M^\bullet, N^\bullet$; define

$$\mathrm{Hom}^n_{C(k)}(M^\bullet, N^\bullet) = \prod_{i \in \mathbb{Z}} \mathrm{Hom}_{k\text{-Mod}}(M^i, N^{i+n}) \quad \text{and} \quad d_H(f) = d_N \circ f - (-1)^n f \circ d_M$$

   Here there is an obvious functor which sends objects to themselves, and sends $\mathrm{Hom}^\bullet \longmapsto H^0(\mathrm{Hom}^\bullet)$. We denote the image of this functor by $H^0(C_{dg}(k))$, and this agrees with the "classical" homotopy category $\mathcal{K}(k)$.

   Similarly we can also define $Z^0(-)$ applied to a dg-category by a near identical rule. Then in our example $Z^0(C_{dg}(k)) = C(k)$, and so sometimes $Z^0(\mathcal{D})$ is called the "underlying category" of $\mathcal{D}$.

4) Denote by $dgcat(k)$ the category of dg-categories and dg-functors. This category has a monoidal structure given by the tensor product of two dg-categories:

$$A \otimes B = \begin{cases} Ob(A\otimes B) = Ob(A) \times Ob(B) & (\text{small cats!}) \\ Hom_{A\otimes B}((a_1,b_1),(a_2,b_2)) = Hom_A(a_1,a_2) \otimes Hom_B(b_1,b_2). \end{cases}$$

In particular this allows us to speak of dg-bimodules. Similar to the above, a dg-$\mathcal{A}\mathcal{B}$-bimodule is a functor $\mathcal{A} \otimes \mathcal{B}^{op} \longrightarrow C_{dg}(k)$.

Def: A dg-functor $F: A \to B$ is called a *quasi-equivalence* if:

- the morphism on $Hom^\bullet_A(a,a') \longrightarrow Hom^\bullet_B(F(a),F(a'))$ is a quasi-isomorphism,
- the induced functor $H^\circ(F): H^\circ(A) \longrightarrow H^\circ(B)$ is essentially surjective.

We in particular are interested in dg-categories up to quasi-equivalence. Hence (ignoring set theoretic issues) we denote by $Ho(dgcat(k))$ the (Gabriel-Zisman) localization of $dgcat(k)$ along quasi-equivalences. However this category is badly behaved and difficult to describe, so we need a better description.

Def: Let M be a category with arbitrary limits and colimits. A *closed model structure* on M consists of the datum of three sets of morphisms in M, the fibrations F, the cofibrations C, and the weak equivalences W; satisfying:

1) $X \xrightarrow{f} Y \xrightarrow{g} Z$ in M, W has a "2 out of 3 rule" (i.e. saturated),
2) If f, g are morphisms in M such that f is a retract of g, that is, there is a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\ A' \arrow[r] & B' \arrow[r] & A' \end{tikzcd}$$

where the horizontal compositions are identities. Then if $g \in F, C,$ or W, so is f.

3) Let

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"] & Y \end{tikzcd}$$

be a commutative square with $i \in C$, $p \in F$. Then if either f or g are in W, then there is a morphism $h: B \to X$ making the triangles commute.

4) Any morphism $f: X \to Y$ in M can be factored in two ways as $f = p \circ i$ and $f = q \circ j$ with $p \in F$, $i \in C \cap W$, $q \in F \cap W$, and $j \in C$.

By definition, the homotopy category of a model category $Ho(M)$ is the localization $M[W^{-1}]$. The existence of a model structure has rather significant implications for the localized category. Indeed two morphisms $f, g: X \to Y$ are called *homotopic* if there exists a diagram

$$\begin{tikzcd} X \arrow[d, "i"'] \arrow[dr, "f"] & \\ C(X) \arrow[r, "h"] & Y \\ X \arrow[u, "j"] \arrow[ur, "g"'] & \end{tikzcd}$$

s.t.

1) there is $p: C(X) \to X$ in $F \cap W$ s.t. $p \circ i = p \circ j = id_X$
2) the induced morphism $X \amalg X \longrightarrow C(X)$ is in C.

Now the axioms of a model category are such that f is homotopic to g in $\mathcal{M}$, then $f = g$ in $Ho(\mathcal{M})$, further, the image of p is an isomorphism, and so are the images of i and j.

<u>Example:</u> On $C(k)$ there is a model structure given by

$F = \{$ surjective morphisms $\}$
$C = \{$ injective morphisms $\}$
$W = \{$ quasi-isomorphisms $\}$.

Then it is known that $Ho(C(k)) \cong D(k)$, the derived category of k-Mod.

If $\mathcal{A}$ is a dg-category, one can give a model structure on (right) $\mathcal{A}$-modules $C(\mathcal{A})$, where the weak-equivalences are the quasi-isomorphisms. The resulting homotopy category is denoted $D(\mathcal{A})$, the derived category of $\mathcal{A}$ (NOT the same as the dg-derived category of a dg-category).

<u>Theorem:</u> $dgcat(k)$ carries a (cofibrantly generated) model structure. The weak equivalences are the quasi-equivalences and fibrations are dg-functors $F : \mathcal{A} \to \mathcal{B}$ such that

1) Morphisms $Hom_{\mathcal{A}}^{i}(c,d) \longrightarrow Hom_{\mathcal{B}}^{i}(F(c), F(d))$ are surjective,
2) for each isomorphism $[g] : F(x) \to F(y)$ in $H^0(\mathcal{B})$, there is an isomorphism $[f] : x \to y$ with $F([f]) = [g]$.

We call the resulting homotopy category Hqe. A model category with a zero object is called a pointed model category:

$$\begin{tikzcd} A \arrow[r] \arrow[dr, hook, "\sim", "C \cap W"'] & 0 \\ & B \arrow[u, "\in F"'] \end{tikzcd}$$

by our assumption on weak factorization. Objects whose unique map to zero is a fibration are called fibrant. For example, all objects in $C(k)$ are fibrant. The final object in dg-cat is the one-point category whose endomorphism complex is the zero complex. Again, everything is fibrant.

Take again $C^{+}(\mathcal{A})$, $\mathcal{A}$ w/ enough injectives (abelian). The weak equivs + cofibrations are the same, but we can ask fibrations to be injective (degreewise) with injective Ker. There is also a dual notion for $C^{-}(\mathcal{A})$ w/ $\mathcal{A}$ enough projectives. Then fibrant (cofibrant) replacement is similar to injective (projective) resolutions.

Dually:

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[dr, hook] & B \\ & C \arrow[u, "\sim"'] \end{tikzcd}$$

So we can replace every element of our category with something which is fibrant and cofibrant, C. The lesson here is that model categories allow one to work only with "good" objects, instead of with all of $Ho(\mathcal{M})$.

<u>Thm:</u> Let $\tilde{\mathcal{M}}$ be the subcategory of bifibrant objects in $\mathcal{M}$. Then $\tilde{\mathcal{M}}[W^{-1}] \xrightarrow{\sim} \mathcal{M}[W^{-1}]$.

Now if $\mathcal{M}$ has coproducts and $A \in \mathcal{M}$, we have a canonical map $A \amalg A \to A$.

$$\begin{tikzcd} A \amalg A \arrow[r] \arrow[dr, hook, "?"'] & A \\ & Cyl(A) \arrow[u] \end{tikzcd}$$

Then as above we have a notion of homotopy:

$$\left.\begin{tikzcd} A \arrow[d] \arrow[dr, "f"] & \\ Cyl(A) \arrow[r] & B \\ A \arrow[u] \arrow[ur, "g"'] & \end{tikzcd}\right\}$$

$f \sim g$. Hence $\tilde{M}/\sim \;\cong\; \tilde{M}[W^{-1}]$.

Def: A model category M is cofibrantly generated if $\exists$ sets I, J of morphisms in M s.t.

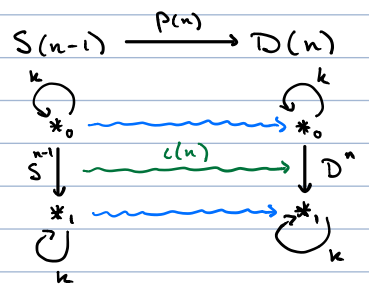
1) All cofibrations are retracts of relative I-complexes, that is, a complex constructed similarly to CW complexes (a colim of iterated pushouts).
2) All acyclic (cofibrations which are weak equivs) are — " — of relative J-complexes.
3) I & J satisfy a small objects argument (don't worry about it).

Ok, so now lets get back on track and discuss the generating cofibrations for the model structure on $dgcat(k)$. We need to give two sets of morphisms I, called the generating cofibrations, and J, the generating trivial cofibrations.

Def: We first define two dg k-module's (= chain complex of k-modules) as follows:

1) $S^n = k[-n]$
2) $D^n = Cone(S^n \xrightarrow{id} S^n)$

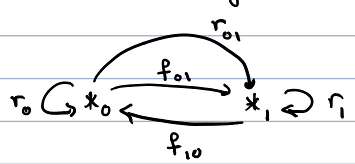
Now consider the categories $S(n-1)$ and $D(n)$, as well as the functor $p(n): S(n-1) \to D(n)$ defined as follows:



$$\begin{tikzcd} S^{n-1} \arrow[r, "\iota(n)"] & D^n \\ \vdots & \vdots \\ 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] \arrow[d] & k \arrow[d, equal] \\ k \arrow[r, equal] \arrow[d] & k \arrow[d] \\ 0 \arrow[r] & 0 \\ \vdots & \vdots \end{tikzcd}$$

Then $I = \{p(n) \mid n \in \mathbb{Z}\} \cup \{\emptyset \to k\}$. ← empty dg-cat.

Def: Let $\alpha(n): k \amalg k \to D(n)$ be the functor sending $*_0$ to $*_0$ and $*_1$ to $*_1$. Now consider the category $\mathcal{K}$ generated by morphisms:



s.t. $f_{ij} \in Hom^0(*_i, *_j)$, $r_i \in Hom^{-1}(*_i, *_i)$, $r_{ij} \in Hom^{-2}(*_i, *_j)$ and $d(f_{ij}) = 0$, $d(r_i) = f_{ji} f_{ij} - \mathrm{Id}_i$, and $d(r_{ij}) = f_{ij} r_i - r_j f_{ij}$.

Then $J = \{\alpha(n) \mid n \in \mathbb{Z}\} \cup \{k \to \mathcal{K} \ (* \to *_0)\}$

The reason for this last (weird) choice of functor in J, is that one has a bijection between dg-functors $\mathcal{K} \to \mathcal{A}$, $\mathcal{A}$ a dg-cat, and pairs (f, c) with f a morphism in $Z^0(\mathcal{A})$ and c a contracting homotopy of cone $(Hom_{\mathcal{A}}(-, f))$. Indeed given a dg-functor $\mathcal{K} \to \mathcal{A}$, the image of f_{01} belongs to $Z^0(\mathcal{A})$, call it f.

Now $f: x \to y$ induces $\hat{f}: Hom(-, x) \to Hom(-, y)$ of Yoneda dg $\mathcal{A}$-modules in $C_{dg}(\mathcal{A})$. Then $Cone(\hat{f}) = \hat{y} \oplus \hat{x}[1]$. A contracting homotopy then must satisfy all the relations as in $\mathcal{K}$, so some minor work proves the claim.

Recall that the homotopy category of $dgcat(k)$ w.r.t. this model structure was denoted $Hqe(k)$ (for "Homotopy w.r.t. quasi-equivalences"). As we discussed, there is a notion of cofibrant replacement $\mathcal{A}_{cof} \xrightarrow[q.e.]{\sim} \mathcal{A}$ which is the identity on objects (proof apparently requires the small objects argument).

Given two dg-categories $\mathcal{A}$ and $\mathcal{B}$, we define $\mathcal{A} \overset{L}{\otimes} \mathcal{B}$ by $\mathcal{A}_{cof} \otimes \mathcal{B}$. When $\mathcal{A}$ is already cofibrant (or k-flat, i.e., all dg k-modules $Hom_{\mathcal{A}}(x,y)$ are k-flat) then there is a quasi-equivalence $\mathcal{A} \overset{L}{\otimes} \mathcal{B} \xrightarrow{\sim} \mathcal{A} \otimes \mathcal{B}$.
If $\mathcal{B}$ is a dg-category, a right dg $\mathcal{B}$-module M ($\mathcal{B}^{op} \to C_{dg}(k)$) is called quasi-representable if it is isomorphic in $D(\mathcal{B})$ to $\hat{y} = Hom(-,y)$ for some $y \in \mathcal{B}$.

<u>Def:</u> Given two dg-cats $\mathcal{A}$ and $\mathcal{B}$, we denote by $rep_{qe}(\mathcal{A},\mathcal{B})$ the full triangulated subcategory of $D(\mathcal{A} \overset{L}{\otimes} \mathcal{B})$ consisting of right dg $\mathcal{A} \overset{L}{\otimes} \mathcal{B}$-modules M s.t. $\forall\, a \in \mathcal{A}$, $M(a,-)$ is quasi-representable

<u>Thm (Toën):</u> $Hom_{Hqe(k)}(\mathcal{A},\mathcal{B}) \cong Iso\ rep_{qe}(\mathcal{A},\mathcal{B})$. Where "Iso" means isomorphism classes.

There will be another model structure on $dgcat(k)$ which we will prefer, but already this lets us do quite a bit, and they will be "built" from this one.

<u>Pretriangulated dg-categories and enhancements</u>

Lets take a moment and recall our original goal. If this is theory is to apply to "commutative" algebraic geometry, we should mention how exactly we take a projective variety X to a dg-category.

So that we do not lose anything, we should require that $H^0(\mathcal{D})$ "recovers" X, in the sense that it should recover $D^b(X)$ (really, $\mathrm{perf}\, X$). But this is a triangulated category, and it's certainly not the case that the homotopy category of a dg-category is always triangulated. What we require is some condition on $\mathcal{D}$ so that $H^0(\mathcal{D})$ is triangulated.

<u>Def:</u> We say a dg category $\mathcal{A}$ is pretriangulated if the image of the Yoneda functor $Z^0(\mathcal{A}) \to C(\mathcal{A})$ sending $a \mapsto Hom_{\mathcal{A}}(-,a) = \hat{a}$ is stable under shifts and extensions. That is, for all $a,b \in \mathcal{A}$ and $n \in \mathbb{Z}$ and any $f: B \to A$, $Cone(\hat{f}) = \widehat{Cone(f)}$ and $\hat{B}[n] = \widehat{B[n]}$.

For us, its enough to know that if $\mathcal{A}$ is pretriangulated, then $H^0(\mathcal{A})$ has the structure of a triangulated category. Further, given a dg-category $\mathcal{A}$, there is a construction which returns a pretriangulated category, $\text{pre-tr}(\mathcal{A})$, called the pretriangulated envelope. To construct this, one proceeds roughly as follows

1) form $\mathbb{Z}(\mathcal{A})$, the dg-category whose objects are pairs (a,r), $a \in \mathcal{A}$, $r \in \mathbb{Z}$, and morphism spaces are $Hom_{\mathbb{Z}(\mathcal{A})}((a,r),(b,s)) = Hom_{\mathcal{A}}(a,b)[s-r]$. Very roughly, we are endowing $\mathcal{A}$ with a "shift" (or suspension) functor.
2) Now define $\text{pre-tr}(\mathcal{A})$ to be the dg-category with objects finite sequences $(x_1,r_1),\dots,(x_n,r_n)$ of objects in $\mathbb{Z}(\mathcal{A})$, together with matrices $M = (m_{ij})$ of morphisms $m_{ij} \in Hom_{\mathbb{Z}(\mathcal{A})}((x_j,r_j),(x_i,r_i))$ such that $m_{ij} = 0$ for $i \geq j$ (strictly upper triangular) and $d(m_{ij}) + \sum_\ell m_{i\ell} m_{\ell j} = 0$.

   Now morphisms in $Hom_{\text{pretr}(\mathcal{A})}(\{(x_i,r_i),M\},\{(y_j,s_j),N\})$ are given by matrices $f = (f_{ij})$ w/ $f_{ij} \in Hom_{\mathbb{Z}(\mathcal{A})}((x_j,r_j),(y_i,s_i))$. The differential of a homogeneous morphism is

$$df = \underbrace{d_{\mathbb{Z}(\mathcal{A})} f}_{\text{applied termwise}} + Nf - (-1)^n f M.$$

Intuitively we are adding all cones, cones of morphisms of cones, etc.

Def: Let T be a triangulated category. A dg enhancement is a pair, $(\mathcal{T}, \varepsilon)$, where $\mathcal{T}$ is a pretriangulated dg-category and $\varepsilon: H^0(\mathcal{T}) \to T$ is a triangulated equivalence. We say two enhancements $\mathcal{T}$ and $\mathcal{T}'$ are equivalent if there is a dg-functor $\phi: \mathcal{T} \to \mathcal{T}'$ inducing an equivalence $H^0(\mathcal{T}) \xrightarrow{\sim} H^0(\mathcal{T}')$. They are strongly equivalent if ϕ can be chosen so that $\varepsilon' \circ H^0(\phi) \cong \varepsilon$.

While the definition is fine, we have yet to construct an example. One may hope that the quotient construction of Verdier lifts to the dg-world, and in fact it does. It is known as the Drinfeld quotient or dg-quotient.

Let $\mathcal{A} \subseteq \mathcal{B}$ be a full dg-subcat of a dg-cat. First we take a "homotopically k-flat" resolution $\tilde{\mathcal{B}} \to \mathcal{B}$. This is a quasi-equivalence $\tilde{\mathcal{B}} \to \mathcal{B}$, and $\tilde{\mathcal{B}}$ is a dg-category whose morphism complexes are homotopically k-flat, i.e. for all acyclic dg k-modules M, the complex $\mathrm{Hom}_{\tilde{\mathcal{B}}}(x,y) \otimes_k M$ (recall that M is just a chain complex). In our case we can use cofibrant replacement $\mathcal{B}_{cof} \to \mathcal{B}$.

Now form a new category by adding, for each $a \in \tilde{\mathcal{A}} \subseteq \tilde{\mathcal{B}}$ a morphism $c_a : a \to a$ in degree -1 and declare $d(c_a) = \mathrm{id}_a$. This contracts every object in $\tilde{\mathcal{A}}$, and we call the resulting category $\mathcal{B}/\mathcal{A}$. In $Hqe(k)$, we even get a "localization functor" $\mathcal{B} \xrightarrow{Q} \mathcal{B}/\mathcal{A}$.

Now on any quasi-compact and separated scheme X, we define a dg-enhancement of $D(\mathrm{Qcoh}X)$ by the dg-quotient $\mathcal{D}(\mathrm{Qcoh}X) = C_{dg}(\mathrm{Qcoh}X)/Ac_{dg}(\mathrm{Qcoh}X)$. It's not hard to see that $H^0(\mathcal{D}(\mathrm{Qcoh}X))$ is equivalent to $D(\mathrm{Qcoh}X)$. Recall that an object C in a triangulated category T is called compact if $\mathrm{Hom}_T(C,-)$ commutes with arbitrary direct sums. It has been shown that the triangulated subcategory of perfect objects in $D(\mathrm{Qcoh}X)$ is $\mathrm{Perf}X$, and so all of this gives a dg-enhancement of $\mathrm{Perf}\,X$, denoted $\mathrm{Perf}_{dg}X$.

$\mathrm{Perf}_{dg}X$ is really the "right" category to work with for many reasons. For one, it is one of the cases where we actually have a uniqueness result.

Theorem (Lunts-Orlov): If X is quasi-projective over a field, then $\mathrm{Perf}\,X$ and $D^b(\mathrm{coh}X)$ have unique enhancements. If is projective, they are strongly unique

This has been extended this to $D(G)$, where G is any Grothendieck category, and in some situations also to the compact objects $D(G)^c$. See Canonaco - Stellari.

Note that apriori, there is little reason to think a triangulated category has an enhancement, or even if it's unique. For example Spectra has no dg-enhancement, and there are rings (quasi-Frobenius) s.t. $\mathrm{Mod}(R)$ has non-unique enhancements.

Already this should be enough to convince you that $Hqe(k)$ might not be the best category since we have non-unique enhancement in some geometric & algebraic context. So what we should do is learn how to get a category in which they are unique (when they exist). What we could do is work with a notion of "pretriangulated" equivalence, but we actually (for reasons later on) want something stronger.

Morita equivalence

Let $\mathcal{A}$ and $\mathcal{B}$ be two dg-categories and X a right $\mathcal{A}$-$\mathcal{B}$ bimodule (dg $\mathcal{A}^{op}\otimes\mathcal{B}$-module). This is a functor $X: \mathcal{A}^{op}\otimes\mathcal{B} \to C_{dg}(k)$, and hence is the datum of complexes $X(B,A)$, $A\in\mathcal{A}$, $B\in\mathcal{B}$, and for each dg $\mathcal{B}$-module (right) M, we have a functor:

$$G : C(\mathcal{B}) \to C(\mathcal{A}), \quad M \mapsto \left(\mathrm{Hom}_{C_{dg}(\mathcal{B})}(X,M) : A \mapsto \mathrm{Hom}_{C_{dg}(\mathcal{B})}(X(-,A), M)\right)$$

and this has a left adjoint $F(L) = L\otimes_{\mathcal{A}} X$. Now using the model structures on the categories of modules, we can define

$$LF(L) = F(L_{cof}) \quad \text{and} \quad RG(M) = G(M_{fib})$$

Now if $\Phi: \mathcal{A}\to\mathcal{B}$ is a dg-functor, then we can get an $\mathcal{A}\otimes\mathcal{B}^{op}$-module by setting $X(B,A) = \mathrm{Hom}_{C_{dg}(\mathcal{B})}(B, \Phi(A))$ is a dg-bimodule. We then get a functor $LF_{\Phi} : D(\mathcal{A}) \to D(\mathcal{B})$.

Def: A dg-functor $\Phi: \mathcal{A}\to\mathcal{B}$ is said to be a Morita equivalence if $LF_{\Phi}: D(\mathcal{A}) \to D(\mathcal{B})$ is an equivalence

Note every quasi-equivalence is a morita equivalence, and so is the morphism $\mathcal{A} \to \text{pre-tr}(\mathcal{A})$.

Thm (Tabuada): $dgcat(k)$ admits a cofibrantly generated model structure where weak equivalences are Morita equivalences and cofibrations are the same as in $Hqe(k)$. The fibrant objects are the dg-categories whose Yoneda embedding $H^0(\mathcal{A}) \to D(\mathcal{A})^c$ is an equivalence

We write $Hmo(k)$ for the resulting localization. Note that $Hmo(k)$ can be obtained from $Hqe(k)$ by left Bousfield localization, and is a pointed category with zero object $* \cong 1$ (one object + one morphism).